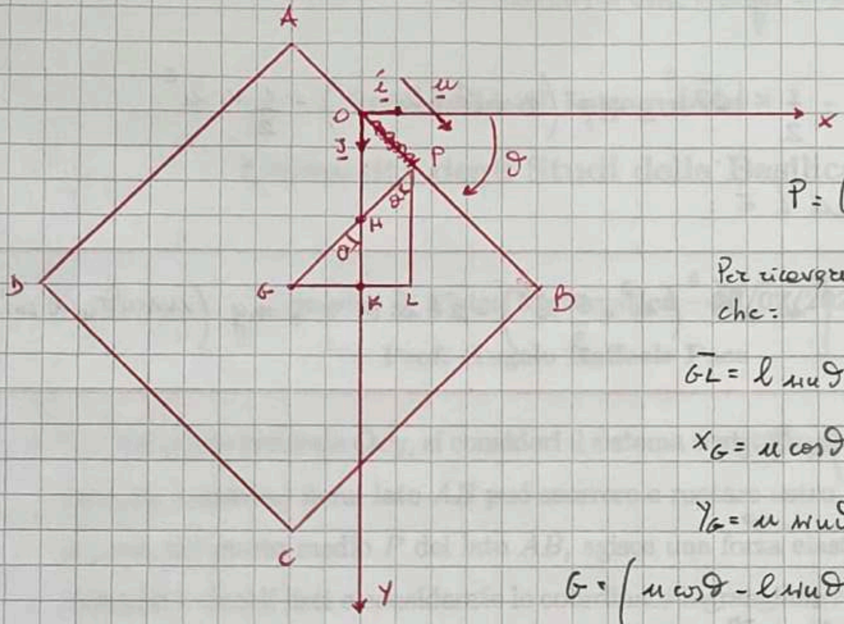
1)

18/02/2026



$P = (u\cos\vartheta ;\ u\sin\vartheta)$

Per ricavare le coordinate di G, osserviamo che:

$\overline{GL} = l\sin\vartheta \Rightarrow \overline{GK} = l\sin\vartheta - u\cos\vartheta \Rightarrow$

$x_G = u\cos\vartheta - l\sin\vartheta$

$\Rightarrow$

$y_G = u\sin\vartheta + l\cos\vartheta$

$G \cdot \left(u\cos\vartheta - l\sin\vartheta ,\ u\sin\vartheta + l\cos\vartheta \right)$

L'energia cinetica è data dal teorema di Konig: $T = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2 \Rightarrow$

$\underline{v}_G = \left(\dot{u}\cos\vartheta - u\dot{\vartheta}\sin\vartheta - l\dot{\vartheta}\cos\vartheta ;\ \dot{u}\sin\vartheta + u\dot{\vartheta}\cos\vartheta - l\dot{\vartheta}\sin\vartheta \right) \Rightarrow$

$v_G^2 = \left(\dot{u}\cos\vartheta - u\dot{\vartheta}\sin\vartheta - l\dot{\vartheta}\cos\vartheta\right)^2 + \left(\dot{u}\sin\vartheta + u\dot{\vartheta}\cos\vartheta - l\dot{\vartheta}\sin\vartheta\right)^2 =$

$= \dot{u}^2\cos^2\vartheta + u^2\dot{\vartheta}^2\sin^2\vartheta + l^2\dot{\vartheta}^2\cos^2\vartheta - 2u\dot{u}\cancel{\dot{\vartheta}\sin\vartheta\cos\vartheta} + 2ul\cancel{\dot{\vartheta}^2\sin\vartheta\cos\vartheta}$

$- 2l\dot{u}\dot{\vartheta}\cos^2\vartheta + \dot{u}^2\sin^2\vartheta + u^2\dot{\vartheta}^2\cos^2\vartheta + l^2\dot{\vartheta}^2\sin^2\vartheta + 2u\dot{u}\cancel{\dot{\vartheta}\sin\vartheta\cos\vartheta}$

$- 2lu\cancel{\dot{\vartheta}^2\sin\vartheta\cos\vartheta} - 2l\dot{u}\dot{\vartheta}\sin^2\vartheta =$

$= \dot{u}^2(\sin^2\vartheta + \cos^2\vartheta) + u^2\dot{\vartheta}^2(\sin^2\vartheta + \cos^2\vartheta) + l^2\dot{\vartheta}^2(\sin^2\vartheta + \cos^2\vartheta)$

$- 2l\dot{u}\dot{\vartheta}(\sin^2\vartheta + \cos^2\vartheta) =$

$= \dot{u}^2 + u^2\dot{\vartheta}^2 + l^2\dot{\vartheta}^2 - 2l\dot{u}\dot{\vartheta}$

Il momento d'inerzia della lamina quadrata rispetto all'asse ortogonale al piano Oxy e passante per G è: $I_G = \frac{2}{3} m l^2$. Quindi:

$T = \frac{1}{2} m \left[\dot{u}^2 + \dot{\vartheta}^2(u^2 + l^2) - 2l\dot{u}\dot{\vartheta} \right] + \frac{1}{3} m l^2 \dot{\vartheta}^2 =$

$= \frac{1}{2} m \left[\dot{u}^2 + \dot{\vartheta}^2\left(u^2 + \frac{5}{3} l^2\right) - 2l\dot{u}\dot{\vartheta} \right]$

Il potenziale delle forze attive agenti sul sistema è:

$$U = U_g + U_e = mg\,y_G - \frac{1}{2}k|\overline{OP}|^2 = mg\,(u\sin\vartheta + l\cos\vartheta) - \frac{1}{2}k u^2$$

Ne viene che la Lagrangiana L è:

$$\boxed{L(\vartheta, u, \dot\vartheta, \dot u) = \frac{1}{2}m\left[\dot u^2 + \dot\vartheta^2\left(u^2 + \frac{5}{3}l^2\right) - 2l\dot u\dot\vartheta\right] + mg\,(u\sin\vartheta + l\cos\vartheta) - \frac{1}{2}k u^2}$$

L'equazione di Lagrange per ϑ è:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\vartheta}\right) - \frac{\partial L}{\partial\vartheta} = 0 \Rightarrow$$

$$\frac{\partial L}{\partial\dot\vartheta} = mu^2\dot\vartheta - ml\dot u \Rightarrow$$

$$\left.\begin{aligned}&\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\vartheta}\right) = m\left(2u\dot u\dot\vartheta + u^2\ddot\vartheta - l\ddot u\right)\\ &\frac{\partial L}{\partial\vartheta} = mg\,(u\cos\vartheta - l\sin\vartheta)\end{aligned}\right\} \Rightarrow$$

$$\boxed{2u\dot u\dot\vartheta + u^2\ddot\vartheta - l\ddot u - gu\cos\vartheta + gl\sin\vartheta = 0}$$

L'equazione di Lagrange per u è:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot u}\right) - \frac{\partial L}{\partial u} = 0 \Rightarrow$$

$$\left.\begin{aligned}&\frac{\partial L}{\partial\dot u} = m\dot u - ml\dot\vartheta \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial\dot u}\right) = m\ddot u - ml\ddot\vartheta\\ &\frac{\partial L}{\partial u} = m\dot\vartheta^2 u + mg\sin\vartheta - ku\end{aligned}\right\} \Rightarrow$$

$$\boxed{m\ddot u - ml\ddot\vartheta - m\dot\vartheta u - mg\sin\vartheta + ku = 0}$$

Le condizioni di equilibrio si ottengono risolvendo il sistema:

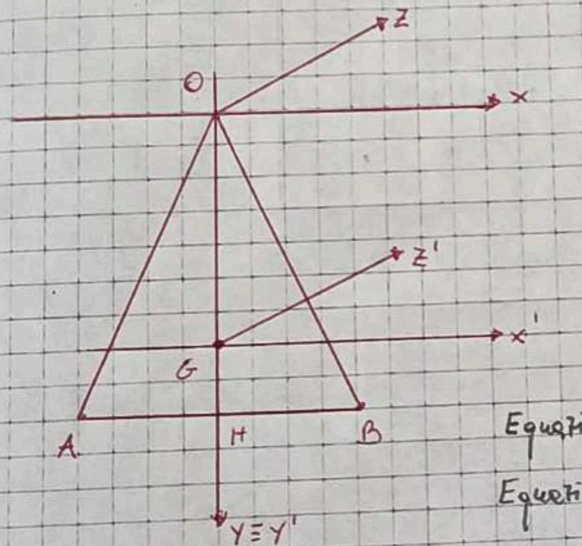
$$\begin{cases}\dfrac{\partial U}{\partial\vartheta} = mg\,(u\cos\vartheta - l\sin\vartheta) = 0\\[2mm] \dfrac{\partial U}{\partial u} = mg\sin\vartheta - ku = 0\end{cases} \Rightarrow \begin{cases}\dfrac{mg}{k}\sin\vartheta\cos\vartheta - l\sin\vartheta = 0\\[2mm] u = \dfrac{mg}{k}\sin\vartheta\end{cases} \Rightarrow$$

$$\begin{cases} \sin\vartheta\left(\dfrac{mg}{k}\cos\vartheta - \ell\right) = 0 \\ u = \dfrac{mg}{k}\sin\vartheta \end{cases} \Rightarrow \begin{cases} \sin\vartheta = 0 \ \vee \ \cos\vartheta = \dfrac{\ell k}{mg} \\ u = \dfrac{mg}{k}\sin\vartheta \end{cases}$$

Posto, per comodità, $\lambda = \dfrac{\ell k}{mg}$, si ha:

$$\boxed{\begin{cases} \vartheta_1 = 0 \\ u_1 = 0 \end{cases} ; \quad \begin{cases} \vartheta_2 = \pi \\ u_2 = \pi \end{cases} ; \quad \begin{cases} \vartheta_3 = \arccos\lambda \\ u_3 = \dfrac{\ell}{\lambda}\sqrt{1-\lambda^2} \end{cases} ; \quad \begin{cases} \vartheta_4 = -\arccos\lambda \\ u_4 = -\dfrac{\ell}{\lambda}\sqrt{1-\lambda^2} \end{cases}}$$

2)



$\overline{AB} = a$, $\overline{OH} = h$

Nel riferimento Oxyz:

$A = \left(-\dfrac{a}{2}; h; 0\right)$ $\quad B = \left(\dfrac{a}{2}; h; 0\right)$

Equazione di $\overline{OA}$: $y = -\dfrac{2h}{a}x$

Equazione di $\overline{OB}$: $y = \dfrac{2h}{a}x$

$$I_x = \frac{2m}{ah}\int_0^h \int_{-\frac{a}{2h}y}^{\frac{a}{2h}y} y^2\, dx\, dy = \frac{2m}{ah}\int_0^h y^2 \left[x\right]_{-\frac{a}{2h}y}^{\frac{a}{2h}y} dy =$$

$$= \frac{2m}{\cancel{a}h}\cdot\frac{\cancel{2}\cancel{a}}{\cancel{2}h}\int_0^h y^3 dy = \frac{2m}{h^2}\left[\frac{y^4}{4}\right]_0^h = \frac{\cancel{2}m}{\cancel{h^2}}\cdot\frac{h^{\cancel{4}\,2}}{\cancel{4}\,2} = \frac{1}{2}mh^2$$

$$I_y = \frac{2m}{ah}\int_0^h \int_{-\frac{a}{2h}y}^{\frac{a}{2h}y} x^2\, dx\, dy = \frac{2m}{ah}\int_0^h \left[\frac{x^3}{3}\right]_{-\frac{a}{2h}y}^{\frac{a}{2h}y} dy =$$

$$= \frac{2m}{ah}\int_0^h \frac{2}{3}\frac{a^3}{8h^3}y^3 dy = \frac{\cancel{2}m}{\cancel{a}h}\cdot\frac{\cancel{2}}{3}\cdot\frac{a^{\cancel{3}\,2}}{\cancel{8}_{2}h^3}\left[\frac{y^4}{4}\right]_0^h =$$

$$= \frac{1}{6}\frac{ma^2}{\cancel{h^4}}\cdot\frac{\cancel{h^4}}{4} = \frac{1}{24}ma^2$$

$$I_z = I_x + I_y = \frac{1}{2} m h^2 + \frac{1}{24} m a^2$$

Poiché il corpo rigido è piano, l'asse Oz è principale d'inerzia $\Rightarrow J_{13} = J_{23} = 0$;

inoltre:

$$J_{12} = \frac{2m}{ah} \int_0^h \int_{-\frac{a}{2h}y}^{\frac{a}{2h}y} xy \, dx dy = \frac{2m}{ah} \int_0^h y \left[\frac{x^2}{2} \right]_{-\frac{a}{2h}y}^{\frac{a}{2h}y} dy = 0$$

Quindi il riferimento Oxyz è principale d'inerzia e:

$$\boxed{\underline{\underline{I}}(O) = \begin{bmatrix} \frac{1}{2} m h^2 & 0 & 0 \\ 0 & \frac{1}{24} m a^2 & 0 \\ 0 & 0 & \frac{1}{2} m h^2 + \frac{1}{24} m a^2 \end{bmatrix}}$$